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FFZ realization of the deformed super Virasoro algebra — Chaichian-Prešnajder type

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Abstract

The q-deformed super Virasoro algebra proposed by Chaichian and Prešnajder is examined. Presented is the realizations by the FFZ algebra (the magnetic translation algebra) defined on a two-dimensional lattice with a supersymmetric Hamiltonian.

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Nearly a decade ago, a series of q-analogues of the Virasoro algebra were investigated through analyzing an infinite set of q-deformed differential operators [1, 2]. Some of these algebras can be organized as a $N = 1$ supersymmetric algebra [3, 4], and there exists a $N = 2$ extension as well [5]. As to the nonsupersymmetric parts, the following things are known. These types of q-deformed Virasoro algebras seem to act as W-infinity algebras on the space of soliton solutions [6], and their decompositions into the FFZ algebra [7] are certainly possible at the level of the differential operator realizations. However, these suggested equivalences are not obvious so far in various observations at the level of field realizations: Sugawara constructions in terms of q-oscillators [3, 8], OPE representations [9, 10], and central extensions [3, 10]. In addition, none of realization-independent map relations is known yet, and it is important to examine relations between various realizations. An interesting remark is that one of these deformed algebras [1]-[3] is certainly a special case of the other (quantum) deformed Virasoro algebra emerged from the context of a lattice model [11].

In this paper, we study the deformed super Virasoro algebra (Chaichian-Prešnajder type) [3] from a bit different point of view. Apart from the above equivalence problem, it is also an interesting question whether or not a supersymmetric extension of the algebra can really match with the concept of a physical (magnetic) deformation as mentioned below. The super algebra [3] consists of the commutation relations (called the algebras q-Vir^F and q-Vir^B in [4]) and the other parts involving supergenerators. In [12], it is shown that the algebra q-Vir^F emerges as a natural generalization of the quantum algebra $\mathcal{U}_q(sl(2))$ in an electron system subjected on a two-dimensional surface in a uniform magnetic field [13, 14]. In this system, rather than the usual translation, the translation accompanied by a gauge transformation factor (the magnetic translation [15]) plays an important role.

A linear combination of the magnetic translations forms the algebra q-Vir^F (with no central extension). This is contrast to the fact that translational invariance (energy-momentum tensor) is related to the Virasoro algebra. Furthermore, it is an interesting framework that a magnetic lattice becomes continuous as a magnetic field vanishes and

then the $q = 1$ case (Virasoro algebra) recovers in this limit. It is curious to examine whether or not this similarity would hold in a supersymmetric case, and hence we construct a couple of realizations of the supersymmetric extension of q-Vir^F and q-Vir^B in terms of the magnetic translation operators. Note that we only deal with centerless algebras, since the magnetic translations are differential operators.

The magnetic translations defined on a two-dimensional lattice (k, n); $k, n \in \mathbf{Z}$, satisfy the relation

$$T_{(k,n)} T_{(l,m)} = q^{\frac{ln-mk}{2}} (q - q^{-1})^{-1} T_{(k+l,n+m)} \ , \tag{1}$$

with realizing the FFZ algebra [7]

$$[\, T_{(k,n)} \, , T_{(l,m)} \,] = [\frac{ln - mk}{2}]_q T_{(k+l,n+m)} \ , \tag{2}$$

where

$$[x]_q = (q^x - q^{-x})/(q - q^{-1}) \ . \tag{3}$$

These relations are also appeared in the recent studies of non-commutative field theory [16]. Hereafter, for the generality of discussion, we assume that the $T_{(k,n)}$ are defined in an abstract sense.

<u>The algebra q-Vir^F</u>

The algebra q-Vir^F is defined by

$$[\, F_n^{(k)} \, , F_m^{(l)} \,] = \frac{1}{2} \sum_{\varepsilon,\eta=\pm 1} [\frac{\varepsilon nl - \eta mk}{2}]_q \frac{[\varepsilon k + \eta l]_q}{[k]_q [l]_q} \, F_{n+m}^{(\varepsilon k + \eta l)} \ , \tag{4}$$

which is the maximal symmetric form in the generator indices [6]. The upper and lower indices on $F_n^{(k)}$ take all integers, however for later convenience, we may exclude $k = 0$ without any contradiction. The central extension of this algebra can be realized by the Sugawara construction of fermionic oscillators [3]. If we assume the relation

$$F_n^{(k)} = F_n^{(-k)} \ , \tag{5}$$

the above algebra reduces to the following form:

$$[\, F_n^{(k)}\, , F_m^{(l)}\,] = \sum_{\varepsilon=\pm 1} [\frac{\varepsilon nl - mk}{2}]_q \frac{[k+\varepsilon l]_q}{[k]_q[\varepsilon l]_q}\, F_{n+m}^{(k+\varepsilon l)} \ . \tag{6}$$

If we consider the $q \to 1$ limit with assuming

$$F_n^{(k)} \quad \to \quad L_n \ , \tag{7}$$

the algebra q-Vir^F becomes the Virasoro algebra

$$[\, L_n\, , L_m\,] = (n-m) L_{n+m} \ . \tag{8}$$

There are two magnetic translation operator realizations for the q-Vir^F generators; one is [12]

$$F_n^{(k)} = \frac{1}{[k]_q} \sum_{\varepsilon=\pm 1} \varepsilon\, T_{(\varepsilon k, n)} \ , \qquad (k \neq 0) \tag{9}$$

and the other is merely given by interchanging the roles of the two components of lattice coordinates (n, m);

$$F_n^{(k)} = \frac{-1}{[k]_q} \sum_{\varepsilon=\pm 1} \varepsilon\, T_{(n, \varepsilon k)} \ , \qquad (k \neq 0) \ . \tag{10}$$

Identifying

$$J_m^{(l)} = T_{(l,m)} \quad \text{or} \quad T_{(m,l)} \qquad\quad \text{for Eqs.(9)} \quad \text{or} \quad (10) \ , \tag{11}$$

the following relation is satisfied in each case:

$$[\, F_n^{(k)}\, , J_m^{(l)}\,] = \frac{1}{[k]_q} \sum_{\varepsilon=\pm 1} \varepsilon\, [\frac{nl - \varepsilon mk}{2}]_q J_{n+m}^{(\varepsilon k + l)} \ . \tag{12}$$

This represents an analogue of the commutation relation between $u(1)$ currents and the Virasoro generators

$$[\, L_n\, , J_m\,] = -m J_{n+m} \ . \tag{13}$$

Here we put a remark. In the realization of q-Vir^F by ghost oscillators, there exists the following closed algebra [10] (in addition to Eq.(6)):

$$[\, F_n^{(k)}\, , R_m^{(l)}\,] = \frac{1}{[k]_q[l]_q} \sum_{\varepsilon=\pm 1} [\frac{\varepsilon nl - mk}{2}]_q [k+\varepsilon l]_q\, R_{n+m}^{(k+\varepsilon l)} \ , \tag{14}$$

$$[\, R_n^{(k)}\, , R_m^{(l)}\,] = \frac{1}{[k]_q [l]_q} \sum_{\varepsilon=\pm 1} [\frac{\varepsilon n l - m k}{2}]_q [k + \varepsilon l]_q \, F_{n+m}^{(k+\varepsilon l)} \ . \tag{15}$$

In the present case, we can realize the generators $R_n^{(k)}$ as

$$R_n^{(k)} = \frac{1}{[k]_q} \sum_{\varepsilon=\pm 1} T_{(\varepsilon k, n)} \ , \qquad (k \neq 0) \ . \tag{16}$$

`The algebra` q-Vir^B

The other counterpart (bosonic) algebra is the algebra q-Vir^B [2, 3]:

$$[\, B_n^{(k)}\, , B_m^{(l)}\,] = \frac{1}{2} \sum_{\varepsilon, \eta=\pm 1} [\frac{n(\varepsilon l + 1) - m(\eta k + 1)}{2}]_q B_{n+m}^{(\varepsilon k + \eta l + \varepsilon \eta)} \ . \tag{17}$$

In contrast to q-Vir^F, the central extension of this algebra can be realized by the Sugawara construction of bosonic oscillators [3]. Note that there are two ways of taking the $q \to 1$ limit:

$$B_n^{(k)} \quad \to \quad L_n \ , \tag{18}$$

$$B_n^{(k)} \quad \to \quad k\, L_n \ , \tag{19}$$

where both limits satisfy the Virasoro algebra (8).

We here present the following four magnetic translation operator realizations (let them referred to as $\mathcal{R}_1^{\pm}$ and $\mathcal{R}_2^{\pm}$):

$$\mathcal{R}_1^{\pm} \ : \quad B_n^{(k)} = \frac{1}{2} \sum_{\varepsilon, \eta=\pm 1} \varepsilon\, q^{\frac{\pm \varepsilon n}{2}} T_{(\eta k + \varepsilon, n)} \ , \tag{20}$$

$$\mathcal{R}_2^{\pm} \ : \quad B_n^{(k)} = \frac{1}{2} \sum_{\varepsilon, \eta=\pm 1} \eta\, q^{\frac{\pm \varepsilon n}{2}} T_{(\eta k + \varepsilon, n)} \ . \tag{21}$$

The deformed $u(1)$ currents are identified for these realizations as follows:

$$J_m^{(l)} = T_{(\pm l, m)} \qquad \text{for} \quad \mathcal{R}_a^{\pm} \qquad (a = 1, 2) \ , \tag{22}$$

and then the commutation relations with $B_n^{(k)}$ for $\mathcal{R}_1^{\pm}$ turn out to be

$$[\, B_n^{(k)},\, J_m^{(l)}\,] = \frac{1}{2} \sum_{\varepsilon, \eta=\pm 1} \varepsilon\, q^{\varepsilon n/2}\, [\frac{nl - m(\eta k + \varepsilon)}{2}]_q \, J_{n+m}^{(\eta k + l + \varepsilon)} \ , \tag{23}$$

and for $\mathcal{R}_2^{\pm}$,

$$[\, B_n^{(k)},\, J_m^{(l)}\,] = \frac{1}{2}\sum_{\varepsilon,\eta=\pm 1} \eta\, q^{\varepsilon n/2}\, [\frac{nl - m(\eta k+\varepsilon)}{2}]_q\, J_{n+m}^{(\eta k+l+\varepsilon)} \ . \tag{24}$$

When we take the $q \to 1$ limits of these commutators, we have to assume (18) for the realizations $\mathcal{R}_1^{\pm}$, and (19) for the realizations $\mathcal{R}_2^{\pm}$, in order to properly reproduce the correct limit (13). This suggests that the realizations $\mathcal{R}_1^{\pm}$ and $\mathcal{R}_2^{\pm}$ certainly possess a different meaning from each other, although both satisfy the same algebra q-Vir^B.

Superalgebra

In addition to the commutators (4) and (17), a supersymmetric generalization of those deformed algebras consists of the following (anti-) commutation relations [3, 10]:

$$[\, F_n^{(k)}, B_m^{(l)}\,] = 0 \ , \tag{25}$$

$$[\, F_n^{(k)}, G_m^{(l)}\,] = \frac{1}{[k]_q(q-q^{-1})}\sum_{\varepsilon=\pm 1} \varepsilon\, q^{\frac{nl-\varepsilon mk}{2}} G_{n+m}^{(\varepsilon k+l)} \ , \tag{26}$$

$$[\, B_n^{(k)}, G_m^{(l)}\,] = \frac{-1}{2(q-q^{-1})}\sum_{\varepsilon,\eta=\pm 1} \eta\, q^{\frac{-n(l+\eta)+m(\varepsilon k+\eta)}{2}} G_{n+m}^{(\varepsilon k+l+\eta)} \ , \tag{27}$$

$$\{\, G_n^{(k)}, G_m^{(l)}\,\} = 2q^{(nl+mk)/2} B_{n+m}^{(k-l)} + \sum_{\varepsilon=\pm 1} \varepsilon q^{\frac{n(\varepsilon-l)-m(k+\varepsilon)}{2}} [k-l+\varepsilon]_q F_{n+m}^{(k-l+\varepsilon)} \ . \tag{28}$$

This superalgebra was first proposed by Chaichian and Prešnajder [3]. The main issue of this paper is to realize this superalgebra in terms of the operators satisfying (1). It is essential to introduce a fermionic freedom in order to express a superalgebra as usual. We thus use a pair of fermionic oscillators

$$\{\, b\,,\, b^{\dagger}\,\} = 1 \ , \qquad b^2 = (b^{\dagger})^2 = 0 \ . \tag{29}$$

For example, these are realized by the Pauli matrices

$$b = \sigma_x + i\sigma_y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad b^{\dagger} = \sigma_x - i\sigma_y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{30}$$

in the Hamiltonian system of a charged particle confined on a two-dimensional surface:

$$H = \frac{1}{2}(p - eA)^2 + \frac{1}{2}B\sigma_z \ , \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{31}$$

In the following, we only assume the relations (29) for the generality of the argument.

Let us consider the realizations of supersymmetric versions of $F_n^{(k)}$ and $B_n^{(k)}$:

$$F_n^{(k)} = \mathcal{R}(F_n^{(k)}) \otimes b\, b^\dagger \ , \tag{32}$$

$$B_n^{(k)} = \mathcal{R}(B_n^{(k)}) \otimes b^\dagger b \ , \tag{33}$$

where $\mathcal{R}$ stands for a certain realization in the case of the non-supersymmetric algebras. It is obvious that for a given realization $\mathcal{R}$, Eqs.(32) and (33) satisfy the commutation relation (25) as well as each of q-Vir^F and q-Vir^B.

The forms of $G_n^{(k)}$ depend on the choice of realization $\mathcal{R}$. In this paper, we employ the realization (9) as $\mathcal{R}$ for the q-Vir^F part. For the q-Vir^B part, we have four candidates for $\mathcal{R}$, as shown in (20) and (21). However we have found only two realizations, which satisfy the relations (26), (27) and (28). One is for the realization $\mathcal{R}_1^+$,

$$G_n^{(k)} = \sqrt{q - q^{-1}}\Big(\sum_{\varepsilon=\pm 1} \varepsilon q^{\varepsilon n/2} T_{(k+\varepsilon,n)} \otimes b + T_{(-k,n)} \otimes b^\dagger \Big) \ , \tag{34}$$

and the other is for the realization $\mathcal{R}_1^-$,

$$G_n^{(k)} = \sqrt{q - q^{-1}}\Big(T_{(k,n)} \otimes b + \sum_{\varepsilon=\pm 1} \varepsilon q^{-\varepsilon n/2} T_{(\varepsilon-k,n)} \otimes b^\dagger \Big) \ . \tag{35}$$

In summary, we have presented the realizations of the deformed superalgebra given by (4), (17) and (25)-(28). The $\mathcal{R}(\mathcal{F}_n^{(k)})$ is given by Eq.(9), and $\mathcal{R}(\mathcal{B}_n^{(k)})$ is either $\mathcal{R}_1^+$ or $\mathcal{R}_1^-$ (see Eq.(20)), while $G_n^{(k)}$ are realized by Eqs.(34) or (35) respectively. Finally, some remarks are in order.

(i) The magnetic translation operator realizations lead only to the centerless algebras, whereas the normal orderings of q-deformed oscillators in the Sugawara construction lead

to the central extensions [3].

(ii) We have restricted ourselves to discuss the Ramond type generators, $G_n^{(k)}$; $n \in \mathbf{Z}$. However, the present results also apply to the Neveu-Schwarz type ($n \in \mathbf{Z} + 1/2$), if one introduces another set of $T_{(n,k)}$ with half-integral indices like on a dual lattice.

(iii) The above four realizations of q-Vir^B have been classified into two types; the realizations $\mathcal{R}_1^{\pm}$ satisfy the present superalgebra, while $\mathcal{R}_2^{\pm}$ do not. In addition, the former type realizes the commutation relation (23), which is different from (24). The role of the latter type should further be investigated.

(iv) The present superalgebra seems different from possible linear combinations of the super FFZ algebra, which does not assume the bilinear forms (such as $b\,b^{\dagger}$) for the non-superalgebra parts. The difference is clear if comparing with other simpler quantum superalgebra [17].

(v) If one wants to introduce a non-commutativity in the Grassmann space, the ordinary commutation relation (29) should be replaced by deformed Grassmann operators like done in a previous work [18]. However, this will probably lead to a different deformed algebra.

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